# A New Approach to Calculating Spatial Impulse Responses

Jørgen Arendt Jensen

Department of Information Technology, Build. 344,
Technical University of Denmark, DK-2800 Lyngby, Denmark

## Abstract

Using linear acoustics the emitted and scattered ultrasound field can be found by using spatial impulse responses as developed by Tupholme and Stepanishen. The impulse response is calculated by the Rayleigh integral by summing the spherical waves emitted from all of the aperture surface. The evaluation of the integral is cumbersome and quite involved for different aperture geometries. This paper re-investigates the problem and shows that the field can be found from the crossings between the boundary of the aperture and a spherical wave emitted from the field point onto the plane of the emitting aperture. Summing the angles of the arcs within the aperture readily yields the spatial impulse response for a point in space. The approach makes is possible to make very general calculation routines for arbitrary, flat apertures in which the outline of the aperture is either analytically or numerically defined. The exact field can then be found without evaluating any integrals by merely finding the zeros of the either the analytic or numerically defined functions. This makes it possible to describe the transducer surface using an arbitrary number of lines for the boundary. The approach can also be used for finding analytic solutions to the spatial impulse response for new geometries of, e.g. ellipsoidal shape. The approach also makes it easy to incorporate any apodization function and the effect from different transducers baffle mountings. Examples of spatial impulse responses for a shape made from lines bounding the aperture is shown along with solutions for Gaussian apodized round transducer.

## 1 Introduction

The calculation of linear, acoustic fields is most often based on the spatial impulse response approach as suggested by Tupholme and Stepanishen [1], [2]. Here the pulsed pressure field is found from a convolution between the acceleration of the transducer surface and the spatial impulse response. The impulse response has been found for a number of geometries. The solutions arrived at are often complicated, since it involves the evaluation of the Rayleigh surface integral. The response depends on the relative position of the field point and many special cases exist, which makes both the derivation of the solution difficult and the evaluation of the responses cumbersome. This makes it necessary to use computers for evaluating and interpreting the responses, since the formulas do not readily give a useful perception of the solution.

It would be appropriate to arrive at general solutions for any geometry that would be both easy to derive analytically and fast to evaluate with a computer. It should strike a balance, where a general easily applicable solution is found, and then the tedious calculations are left to the computer. This has previously been sought achieved by dividing the aperture surface into smaller elements like rectangles [3] or triangles [4], and then sum the response for the sub-elements. Often the transducer must be divided into many elements and only a piece-wise approximation to the apodization is obtained. The fitting to the actual surface is also only approximative for round or oval surfaces; even when using a triangular shape.

Spatial impulse responses from bounded and non-apodized apertures always have discontinuities due to their sharp edges, which makes it difficult to keep the full energy and spectral content in a sampled evaluation. Various techniques have been applied for coping with the discontinuities in the spatial impulse response. This has included using very high sampling frequencies, making a time adapted evaluation, or using the integrated response. Computer evaluation is, thus, always necessary, when evaluating spatial impulse responses.

This paper therefore suggests a new approach to calculating the spatial impulse response in which the computer is involved at an earlier stage in the evaluation of the responses. The paper shows that the response is determined by the crossings of the boundary of the aperture by the spherical wave emitted from the field point. For flat apertures this observation makes it possible to derive a general approach for calculating the spatial impulse response for any aperture geometry and find the response with no approximation. It is also shown in how an arbitrary apodization can be introduced through a simple one-dimensional integration, and how the solution also can be applied to both the soft baffle and rigid baffle situations.
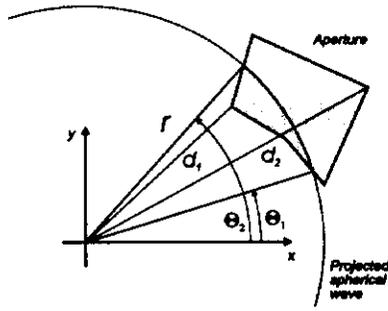
Figure 1: Definition of distances and angles in the aperture plan for evaluating the Rayleigh integral.

## 2 Basic theory

The spatial impulse response is found from the Rayleigh integral given by [2]:

$$h(\vec{r}_1,t) = \int_S \frac{\delta(t - \frac{|\vec{r}_1 - \vec{r}_2|}{c})}{2\pi |\vec{r}_1 - \vec{r}_2|} dS \qquad (1)$$

when the aperture $S$ is mounted in an infinite, rigid baffle. Here $\vec{r}_1$ denotes the position of the field point, $\vec{r}_2$ denotes a position on the aperture, $c$ is the speed of sound, and $t$ is time. The integral is essentially a statement of Huyghen's principle that the field is found by summing the radiated spherical waves from all parts of the aperture. This can also be reformulated, due to acoustic reciprocity, as finding the part of the spherical wave emanating from the field point that intersects the aperture. The task is, thus, to project the field point onto the plane coinciding with the aperture, and then finding the intersection of the projected spherical wave (the circle) with the active aperture.

Rewriting the integral into polar coordinates gives:

$$h(\vec{r}_1,t) = \int_{\Theta_1}^{\Theta_2} \int_{d_1}^{d_2} \frac{\delta(t - \frac{R}{c})}{2\pi R} r\, dr\, d\Theta \qquad (2)$$

where $r$ is the radius of the projected circle and $R$ is the distance from the field point to the aperture given by $R^2 = r^2 + z_p^2$. Here $z_p$ is the field point height above the $x - y$ plane of the aperture. The projected distances $d_1, d_2$ are determined by the aperture and are the distance closest and furthest away from the aperture, and $\Theta_1, \Theta_2$ are the corresponding angles for a given time (see Fig. 1).

Introducing the substitution $2RdR = 2rdr$ gives

$$h(\vec{r}_1,t) = \frac{1}{2\pi} \int_{\Theta_1}^{\Theta_2} \int_{R_1}^{R_2} \delta\left(t - \frac{R}{c}\right) dR\, d\Theta \qquad (3)$$

The variables $R_1$ and $R_2$ denotes the edges closest and furthest away from the field point. Finally using the substitution $t' =$

$R/c$ gives

$$h(\vec{r}_1,t) = \frac{c}{2\pi} \int_{\Theta_1}^{\Theta_2} \int_{t_1}^{t_2} \delta(t - t') dt'\, d\Theta \qquad (4)$$

For a given time instance the contribution along the arc is constant and the integral gives

$$h(\vec{r}_1,t) = \frac{\Theta_2 - \Theta_1}{2\pi} c \qquad (5)$$

when assuming the circle arc is only intersected once by the aperture. The angles $\Theta_1$ and $\Theta_2$ are determined by the intersection of the aperture and the projected spherical wave, and the spatial impulse response is, thus, solely determined by these intersections, when no apodization of the aperture is used. The response can therefore be evaluated by keeping track of the intersections as a function of time.

## 3 A new approach

From the derivation in the last section it can be seen that the spatial impulse response in general can be expressed as

$$h(\vec{r}_1,t) = \frac{c}{2\pi} \sum_{i=1}^{N(t)} \Theta_2^{(i)}(t) - \Theta_1^{(i)}(t) \qquad (6)$$

where $N(t)$ is the number of arc segments that crosses the boundary of the aperture for a given time and $\Theta_2^{(i)}(t), \Theta_1^{(i)}(t)$ are the associated angles of the arc. The calculation can, thus, be formulated as finding the angles of the aperture edge's intersections with the projected spherical wave, sorting the angles, and then sum the arc angles that belong to the aperture. Finding the intersections can be done from the description of the edges of the aperture. A triangle can be described by three lines, a rectangle by four, and the intersections are then found from the intersections of the circle with the lines. This makes it possible to devise a general approach for calculating spatial impulse responses for any flat, bounded aperture, since the task is just to find the intersections of the boundary with the circle.

The spatial impulse response is calculated from the time the aperture first is intersected by a spherical wave to the time for the intersection furthest away. The intersections are found for every time instance and the corresponding angles are sorted. The angles lie in the interval from 0 to $2\pi$. It is then found whether the arc between two angles belongs to the aperture, and the angle difference is added to the sum, if the arc segment is inside the aperture. This yields the spatial impulse response according to Eq. (6). The approach can be described by the pseudo-algorithm:

```
t = d1/c;
while t < d2/c
```

```
r = c*t;
{Th,N} = find_intersec_angles(r,boundary);
sort(Th);
sum = 0;
for i=1 to N-1
    if inside_bndry(point((Th[i+1]+T[i])/2))
        sum = sum + (Th[i+1]-Th[i])/(2*pi)*c;
        i = i + 1;
    end if;
    end for i;
if inside_bndry(point(2*pi-Th[N]+Th[1])/2)
    sum = sum + (2*pi-Th[N]+Th[1])/(2*pi)*c;
    end if;
h[t] = sum;
t = t + 1/fs;
end while;
```



Figure 2: Definition of angle used for a soft baffle.

Note that $i$ is increased by one whenever a point inside the boundary is meet, since the next part of the arc must have crossed an edge, and therefore is outside the active aperture.

The only part of the algorithm specific to the aperture is the determination of the intersections and the whether the point is inside the aperture.

# 4 Apodization and soft baffle

Often ultrasound transducers do not vibrate as a piston over the aperture. This can be due to the clamping of the active surface at its edges, or intentionally to reduce side-lobes in the field. Applying for example a Gaussian apodization will significantly lower side lobes and generate a field with a more uniform point spread function as a function of depth. Apodization can be introduced in (2) by writing

$$h(\vec{r}_1,t) = \int_{\Theta_1}^{\Theta_2} \int_{d_1}^{d_2} a_p(r,\Theta) \frac{\delta(t-\frac{R}{c})}{2\pi R} r \, dr \, d\Theta \quad (7)$$

in which $a_p(r,\Theta)$ is the apodization over the aperture. Using the same substitutions as before yields

$$h(\vec{r}_1,t) = \frac{c}{2\pi} \int_{\Theta_1}^{\Theta_2} \int_{t_1}^{t_2} a_{p1}(t',\Theta)\delta(t-t') \, dt' \, d\Theta \quad (8)$$

where $a_{p1}(t',\Theta) = a_p(\sqrt{(ct')^2 - z_p^2}, \Theta)$. The inner integral is a convolution of the apodization function with a $\delta$-function and readily yields

$$h(\vec{r}_1,t) = \frac{c}{2\pi} \int_{\Theta_1}^{\Theta_2} a_{p1}(t,\Theta) \, d\Theta \quad (9)$$

The response for a given time point can, thus, be found by integrating the apodization function along the fixed arc with a radius of $r = \sqrt{(ct)^2 - z_p^2}$ for the angles for the active aperture. Any apodization function can therefore be incorporated into the calculation by employing numerical integration.

Often the assumption of an infinite rigid baffle for the transducer mounting is not appropriate and another form of the Rayleigh integral must be used. For a soft baffle, in which the pressure on the baffle surface is zero, the Rayleigh-Sommerfeld integral is used. This is [5]

$$h_s(\vec{r}_1,t) = \int_S \frac{\delta(t - \frac{|\vec{r}_1 - \vec{r}_2|}{c})}{2\pi |\vec{r}_1 - \vec{r}_2|} \cos\varphi \, dS \quad (10)$$

Here $\cos\varphi$ is the angle between the line through the field point orthogonal to the aperture plane and the radius of the spherical wave as shown in Fig. 2. The angles $\varphi$ is fixed for a given radius of the projected spherical wave and thus for a given time. It is given by

$$\cos\varphi = \frac{z_p}{R} = \frac{z_p}{ct} \quad (11)$$

The Rayleigh-Sommerfeld integral can then be rewritten as

$$h_s(\vec{r}_1,t) = \cos\varphi \int_S \frac{\delta(t - \frac{|\vec{r}_1 - \vec{r}_2|}{c})}{2\pi |\vec{r}_1 - \vec{r}_2|} \, dS = \frac{z_p}{ct} h(\vec{r}_1,t) \quad (12)$$

and the spatial impulse response can be found from the normal spatial impulse response by multiplying with $z_p/(ct)$.

From this derivation is can be seen that any angular response inbetween that for a rigid and soft baffle mounting readily can be handled by the method through post-multiplication of the standard spatial impulse response by the angular compensation factor.

# 5 Solution for polygons

The boundary of any polygon can be defined by a set of bounding lines as shown in Fig. 3. The active aperture is then defined as lying on one side of the line as indicated by the arrows, and a point on the aperture must be placed correctly in relation to all lines. The test whether a point is on the aperture is thus to go through all lines and test whether the point lies in the active half space for the line, and stop if it is not. The point is inside the aperture, if it passes the test for all lines.
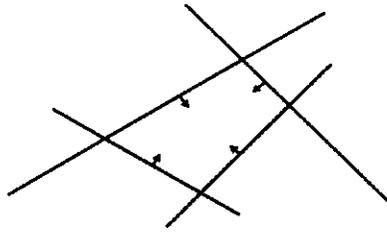
Figure 3: Definition of bounding lines for polygon transducer. The arrows indicates the half-planes for the active aperture.

The intersections are found from the individual intersections between the projected circle and the lines. They are determined from the equations for the projected spherical wave and the line:

$$r^2 = (x - x_0)^2 + (y - y_0)^2$$
$$y = \alpha x + y_1 \qquad (13)$$
$$r^2 = (ct)^2 - z_p^2$$

Here $(x_0, y_0)$ is the center of the circle, $\alpha$ the slope of the line, and $y_1$ its intersect with the $y$-axis. The intersection is given by:

$$0 = (1 + \alpha^2)x^2 + (2\alpha y_1 - 2x_0 - 2y_0\alpha)x$$
$$+ (y_0^2 + y_1^2 + x_0^2 - 2y_0 y_1 - r^2)$$
$$= Ax^2 + Bx + C \qquad (14)$$

with the solution for $x$ given by

$$D = (2B)^2 - 4AC \qquad (15)$$
$$x = \frac{-B \pm \sqrt{D}}{2A} \qquad (16)$$

and the angle is

$$\Theta = \arctan\left(\frac{y}{x}\right) \qquad (17)$$

Intersection between the line and the circle is only found if $D > 0$. A determinant $D < 0$ indicates that the circle intersected the line.

The times for discontinuities in the spatial impulse response are given by the intersections of the lines that defines the aperture's edges and by the minimum distance from the projected field point to the lines. The minimum distance is found from a line passing through the field point that is orthogonal to the bounding line. The intersection between the orthogonal line and the bounding line is:

$$x = \frac{\alpha y_p + x_p - \alpha y_1}{\alpha^2 + 1} \qquad (18)$$
$$y = \alpha x + y_1$$

where $(x_p, y_p, z_p)$ is the position of the field point. For an infinite slope line the solution is $x = x_1$ and $y = y_p$. The corresponding time is:

$$t_i = \frac{\sqrt{(x - x_p)^2 + (y - y_p)^2 + z_p^2}}{c} \qquad (19)$$

The intersections of the lines are also found, and the corresponding times are calculated by (19) and sorted in ascending order. They indicate the start and end time for the response and the time points for discontinuities in the response. Similar solutions can be found for other geometries like circles. A derivation of this can be found in [6].

# 6 Solution for parametric surfaces

For ellipses or other higher order parametric surfaces it is in general not easy to find analytic solutions for the spatial impulse response. The approach described in this paper can however devise a simple solution to the problem, since it has been shown that it is the intersections between the projected spherical wave and the edge of the aperture that uniquely determines the spatial impulse response. It is therefore possible to use root finding for a set of (non-linear) equations for finding these intersections. The problem is to find when both the spherical wave and the aperture have crossing contours in the plane of the aperture, i.e., when

$$(ct)^2 - z_p^2 - (x - x_p)^2 - (y - y_p)^2 = 0 \qquad (20)$$
$$S(x, y) = 0$$

in which $S(x, y) = 0$ defines the boundary of the aperture. The problem of numerically finding these roots is in general not easy, if a good initial guess on the position of the intersections is not found. Good initial values are however found here, since the intersections must lie on the projected circle and the intersections only move slightly from time point to time point. An efficient Newton-Raphson algorithm can therefore be devised for finding the intersections, and the approach detailed here can be made to find the spatial impulse response for any flat transducer geometry with an arbitrary apodization and baffle mounting.

# 7 Examples

The first example is for a complicated aperture, where its bounding lines are shown on the top grah in Fig. 4 and the calculated spatial impulse response is shown in the bottom graph. Responses have been calculated from the center position for $x = 0$ mm, $y = 0$ mm to the position $x = 14$ mm, $y = 0$ mm in increments of 1 mm. The distance to the transducer surface was always 10 mm $(= z)$. A complicated response with a number of discontinuities is seen due to the many edges of the aperture.
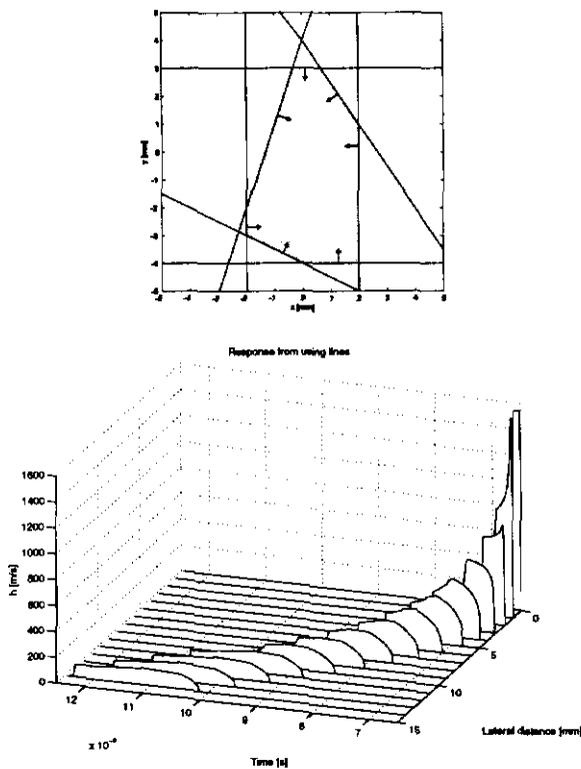
Figure 4: Complex aperture and its spatial impulse response. The top graphs shows the bounding lines defining the complex aperture, and he arrows indicates the half-plane for the active aperture. The bottom graph shows the spatial impulse response from the complex aperture.

The last example shows the response from a circular, flat transducer calculated with the new method. Two different cases are shown in Fig. 5. The top graph shows the traditional spatial impulse response when no apodization is used, so that the aperture vibrates as a piston. The field is calculated 10 mm from the front face of the transducer starting at the center axis of the aperture. Twenty-one responses for lateral distance of 0 to 20 mm off axis are then shown. The same calculation is repeated in the bottom graph, when a Gaussian apodization has been imposed on the aperture. The vibration amplitude is a factor of $1/\exp(4)$ less at the edges of the aperture than at the center. It is seen how the apodization reduces some of the sharp discontinuities in the spatial impulse response.

# References

[1] G. E. Tupholme. Generation of acoustic pulses by baffled plane pistons. *Mathematika*, 16:209–224, 1969.

[2] P. R. Stepanishen. Transient radiation from pistons in an infinte planar baffle. *J. Acoust. Soc. Am.*, 49:1629–1638,
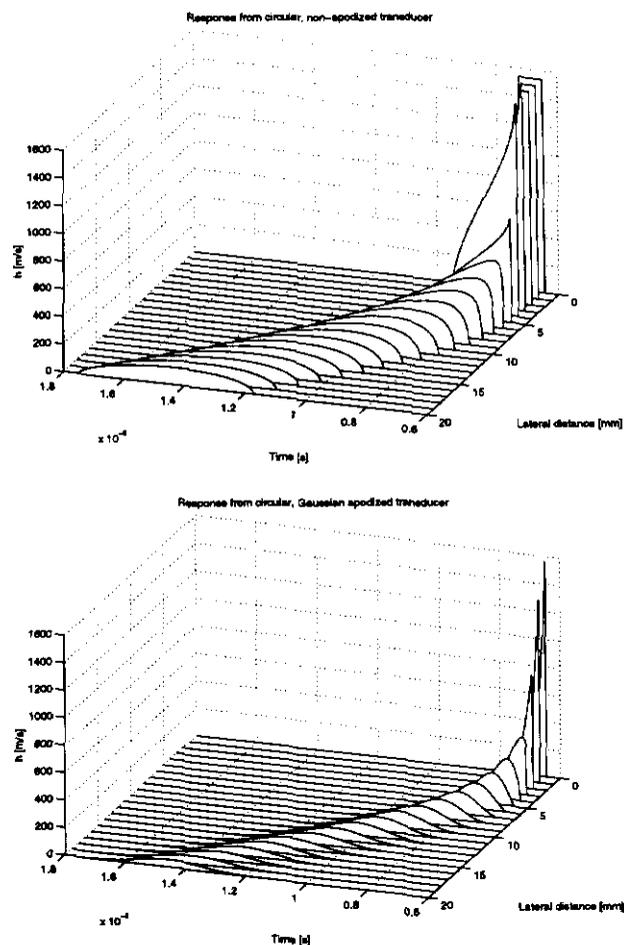


Figure 5: Spatial impulse response from a circular aperture calculated with the new method. Graphs are shown without apodization of the aperture (top) and with a Gaussian apodization function (bottom).

1971.

[3] J. A. Jensen and N. B. Svendsen. Calculation of pressure fields from arbitrarily shaped, apodized, and excited ultrasound transducers. *IEEE Trans. Ultrason., Ferroelec., Freq. Contr.*, 39:262–267, 1992.

[4] J. A. Jensen. Simulating arbitrary geometry ultrasound transducers using triangles. In *Proc. IEEE Ultrason. Symp.*, volume 2, pages 885–888, 1996c.

[5] J. W. Goodman. *Introduction to Fourier optics*. McGraw Hill Inc., New York, second edition edition, 1996.

[6] J. A. Jensen. Calculation of spatial impulse responses in ultrasound. *submitted to JASA*, 1997.